Lecture 2: The discussion above generalizes to Kac-Moody Lie algebras

Def: a generalized Cartan matrix C is an $I \times I$ integer matrix such that

$$\boxed{c_{ij} = \frac{2d_{ij}}{d_{ii}}}, \text{ where } d_{ij} = d_{ji} \in \begin{cases} 2\mathbb{N} & \text{if } i=j \\ \mathbb{Z}_{\leq 0} & \text{if } i \neq j \end{cases}$$

some finite set

We assume $\gcd(d_{ii}) = 2$; C is called symmetric if $d_{ii} = 2$, $\forall i \in I$

Def: to a generalized Cartan matrix C, we associate a

Kac-Moody Lie algebra $\mathfrak{g}$

- generators $\{E_i, F_i, H_i\}_{i\in I}$
- relations $[H_i, H_j] = 0$

$[H_i, E_j] = c_{ij} E_j$

$[H_i, F_j] = -c_{ij} F_j$

$[E_i, F_j] = \delta_{ij} H_i$

$\forall i \neq j$, $\mathrm{ad}_{E_i}^{1-c_{ij}}(E_j) = \mathrm{ad}_{F_i}^{1-c_{ij}}(F_j) = 0$

$\mathfrak{g}$ is called simply-laced if C is symmetric

Drinfeld-Jimbo quantum group $U_q(\mathfrak{g})$

- generators $\{e_i, f_i, k_i\}_{i\in I}$
- relations $k_i k_j = k_j k_i$, $k_i k_i^{-1} = 1$

k_i "=" $q_i^{H_i}$ where $q_i = q^{\frac{d_{ii}}{2}}$

$$\begin{cases} k_i e_j = q^{d_{ij}} e_j k_i \\ k_i f_j = q^{-d_{ij}} f_j k_i \end{cases}$$

$$e_i f_j - f_j e_i = \delta_{ij} \frac{k_i - k_i^{-1}}{q_i - q_i^{-1}}$$

$\forall i \neq j$, $\displaystyle\sum_{k=0}^{1-c_{ij}} \binom{1-c_{ij}}{k}_{q_i} e_i^k e_j e_i^{1-c_{ij}-k} = 0$ and same for f's

↳ appropriate ratio of q_i factorials

Motivation: when $D = (d_{ij})$ is positive definite, $\mathfrak{g} \simeq$ complex semisimple Lie algebra (Serre's theorem)

Terminology: either $\mathfrak{g}$ and $U_q(\mathfrak{g})$ are called

- finite
- affine
- indefinite

type if C has the property that

- D is positive definite
- $D = \bigoplus$ zero det matrices with principal minors positive definite
- all other types

Ex: $\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$

Ex: $\begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$

Ex: $\begin{pmatrix} 2 & -3 \\ -3 & 2 \end{pmatrix}$

Dynkin diagram is also called "finite type", "tame type" and "wild type" in the three cases above: one vertex per $i \in I$.

if $c_{ij} = c_{ji} = -1$ | if $c_{ij} = -2$, $c_{ji} = -1$ | if $c_{ij} = -3$, $c_{ji} = -1$ | otherwise

Ringel - Hall algebras : let A be a hereditary, finitary, abelian category

$\mathrm{Ext}^{\geq 2}$ vanishes

$|\mathrm{Hom}(M,N)| < \infty$
$|\mathrm{Ext}^1(M,N)| < \infty$
$\forall\, M,N \in \mathrm{Ob}(A)$

$\exists$ kernels, cokernels, direct sums, exact sequences etc

Because of these assumptions, the quantity $\chi(M,N) = \dfrac{|\mathrm{Hom}(M,N)|}{|\mathrm{Ext}^1(M,N)|}$ is additive in short exact sequences, i.e.

$\chi(M,N) = \chi(M',N)\,\chi(M'',N)$ if $\exists$ s.e.s. $0 \to M' \to M \to M'' \to 0$
$\quad = \chi(M,N')\,\chi(M,N'')$ if $\exists$ s.e.s $0 \to N' \to N \to N'' \to 0$

Let $\langle M,N\rangle = \sqrt{\chi(M,N)}$
$(M,N) = \langle M,N\rangle\langle N,M\rangle$

In other words, χ descends to the Grothendieck group $K(A) = \bigoplus_{M \in \mathrm{Ob}(A)/\sim} \mathbb{Z}\cdot[M] \Big/ \begin{smallmatrix}[M]=[M']+[M''] \\ \text{if } \exists \text{ s.e.s.} \\ 0\to M'\to M\to M''\to 0\end{smallmatrix}$

<u>Def</u> (Ringel): the Hall algebra of A is

$$H_A = \bigoplus_{M \in \mathrm{Ob}(A)/\sim} \mathbb{C}\cdot[M] = \left\{ \text{functions } f : \mathrm{Ob}(A)/\sim \;\to \mathbb{C} \text{ with finite support} \right\}$$

endowed with the following associative multiplication

$$[M]\cdot[N] = \langle M,N\rangle \sum_{P \in \mathrm{Ob}(A)/\sim} [P]\cdot \frac{\#\{0\to N\to P\to M\to 0\}}{|\mathrm{Aut}(M)|\,|\mathrm{Aut}(N)|} \iff (f\cdot g)(P) = \sum_{K \subseteq P} f(P/K)\,g(K)\,\langle P/K, K\rangle$$

<u>Def</u> (Green): consider the enlarged Hall algebra $\widetilde{H}_A = \dfrac{H_A \otimes \mathbb{C}[k_M]_{M\in K(A)}}{k_M[N] = [N]k_M\cdot (M,N)}$

Then the assignment $\Delta(k_M) = k_M \otimes k_M$
$\Delta(f) = f_1 k_{f_2} \otimes f_2$

(notation: if $0\to M'\to M\to M''\to 0$ then $k_M = k_{M'}k_{M''}$)

where $\boxed{f_1(M)\otimes f_2(N) = \dfrac{\langle M,N\rangle^{-1}}{|\mathrm{Ext}^1(M,N)|} \displaystyle\sum_{P\in \mathrm{Ext}^1(M,N)} f(P)}$

makes $\widetilde{H}_A$ into a bialgebra; moreover, $\widetilde{H}_A \otimes \widetilde{H}_A \xrightarrow{(\cdot,\cdot)} \mathbb{C}$, $\left([M]k_x, [N]k_y\right) = \dfrac{\delta_{MN}}{|\mathrm{Aut}\, M|}(x,y)$
is a bialgebra pairing, i.e. $(\alpha\beta, \gamma) = (\alpha\otimes\beta, \Delta(\gamma))$

<u>Def</u>: a quiver Q is an oriented graph; we denote its vertex set I, and let
$b_{ij} = \#$arrows $i \to j$

a representation of a quiver Q is an assignment
$i \in I \rightsquigarrow$ vector space V_i (assume finite dim)
arrow $i \xrightarrow{e} j \rightsquigarrow$ linear map $V_i \xrightarrow{\phi_e} V_j$

form a category with morphisms
$$\begin{array}{ccc} V_i & \longrightarrow & V_i' \\ \phi_e \downarrow & \square & \downarrow \phi_e' \\ V_j & \longrightarrow & V_j' \end{array}$$

Prop: the category of quiver representations ($\mathrm{Rep}\, Q$) over any field $\mathbb{K}$ is hereditary

↳ also finitary if $\mathbb{K}$ is a finite field

Def: a quiver representation is called nilpotent if $\exists N$ s.t. any composition of any N among the maps ϕ_e is equal to 0 (let $\mathrm{Rep}^{\mathrm{nilp}} Q \subseteq \mathrm{Rep}\, Q$ denote the corresponding full subcat)

Prop: the simple objects in $\mathrm{Rep}^{\mathrm{nilp}} Q$ are $S_i = \{ 0 \overset{0}{\rightleftarrows} \underset{\text{vertex } i}{\mathbb{C}} \overset{0}{\rightleftarrows} 0 \}$

and we have $\mathrm{Hom}(S_i, S_j) \cong \mathbb{K}^{\delta_{ij}}$; moreover $K(\mathrm{Rep}^{\mathrm{nilp}} Q) = \mathbb{Z}^I$

$\mathrm{Ext}^1(S_i, S_j) \cong \mathbb{K}^{b_{ij}}$ is generated by $[S_i]$.

Example: Q = Jordan quiver, any nilpotent representation is a direct sum of $J_n = \{$ n-dimensional vector space with endomorphism $\begin{pmatrix} 0 & 1 & & \\ & 0 & 1 & \\ & & 0 & 1 \\ & & & 0 \end{pmatrix} \}$

Thm (Steinitz): $H_{\mathrm{Rep}^{\mathrm{nilp}} Q} \cong \mathbb{C}[J_1, J_2, \ldots]$ with coproduct given by

$$\Delta(J_n) = \sum_{a+b=n} J_a \otimes J_b \, |\mathbb{K}|^{-ab}$$

technically you should insert k_S^b here, but since $(M,N)=1$ $\forall$ objects M,N, this isn't necessary

$\Downarrow$

$\exists$ isomorphism

$H_{\mathrm{Rep}^{\mathrm{nilp}} Q} \cong$ Symmetric polynomials

$J_n \longrightarrow e_n = \sum_{i_1 < \ldots < i_n} x_{i_1} \cdots x_{i_n}$

For a general quiver Q, we call it
- finite type
- tame
- wild

if $\mathrm{Rep}^{\mathrm{nilp}} Q$ has
- finitely many indecomposables $\forall \mathbb{K}$
- one-parameter families of indecomposables in any dimension
- otherwise

the underlying graph is the Dynkin diagram associated to the generalized Cartan matrix $c_{ij} = 2 - b_{ij} - b_{ji}$

Thm: the three cases above correspond to finite type, affine and indefinite generalized Cartan matrices, respectively

Now assume $\mathbb{K} = \mathbb{F}_{q^2}$ for some $q \in \sqrt{p^N}$

Thm (Gabriel): the assignment $U_q^+(\mathfrak{g}) \longrightarrow H_{\mathrm{Rep}^{\mathrm{nilp}} Q}$ is an algebra homomorphism

it's an isomorphism if and only if Q finite type

$e_i \rightsquigarrow [S_i]$

(extends to $U_q^{\geq}(\mathfrak{g}) \longrightarrow \tilde{H}_{\mathrm{Rep}^{\mathrm{nilp}} Q}$ a bialgebra homomorphism

$k_i \rightsquigarrow k_{S_i}$

Note: $\mathfrak{g}$ only depends on underlying graph of Q